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NONCOMMUTING MAPS AND INVARIANT APPROXIMATIONS

Abstract. We obtain common fixed point results for generalized I-nonexpansive compatible as well as weakly compatible maps. As applications, various best approximation results for this class of maps are derived in the setup of certain metrizable topological vector spaces.

1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real-valued function $\|\cdot\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$,
(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$,
(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$,

for all x, $y \in X$ and all scalars α. The pair $(X, \|, \|_p)$ is called a p-normed space. It is a metric linear space with a translation invariant metric d_p defined by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of the usual normed space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$(see [15]). The spaces l_p and L_p, $0 < p \leq 1$ are p-normed spaces. A p-normed space is not to necessarily a locally convex space. Recall that dual space X^* separates points of X (or equivalently X^* is total [18]) if for each nonzero $x \in X$, there exists $f \in X^*$ such that $f(x) \neq 0$. In this case the weak topology on X is well-defined and is Hausdorff. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0, 1]$, $0 < p < 1$, the space of to the power p integrable functions, or $X = S[0, 1]$, the space of measurable functions, then

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$X^* = \{0\}$(see [15, 18, 20]). However, there are some non-locally convex spaces X (such as the p-normed spaces l_p, $0 < p < 1$) whose dual X^* separates the points of X.

Let X be a metric linear space and M a nonempty subset of X. The set $P_M(u) = \{x \in M : d(x,u) = dist(u,M)\}$ is called the set of best approximants to $u \in X$ from M, where $dist(u,M) = inf\ \{d(y,u) : y \in M\}$. We shall use N to denote the set of positive integers, $cl(S)$ to denote the closure of a set S. The diameter of M is denoted and defined by $\delta(M) = \sup\{\|x-y\| : x, y \in M\}$. A mapping $I : X \to X$ has diminishing orbital diameters (d.o.d.) [13] if for each $x \in X$, $\delta(O(x)) < \infty$ and whenever $\delta(O(x)) > 0$, there exists $n = n_x \in N$ such that $\delta(O(x)) > \delta(O(I^n(x)))$, where $O(x) = \{I^k(x) : k \in N \cup \{0\}\}$ is the orbit of I at x and $O(I^n(x)) = \{I^k(x) : k \in N \cup \{0\}$ and $k \geq n\}$ is the orbit of I at $I^n(x)$ for $n \in N \cup \{0\}$. Let I be a self-map of a topological space X. The orbit $O(x)$ of I at x is proper if and only if $O(x) = \{x\}$ or there exists $n = n_x \in N$ such that $cl(O(I^n(x)))$ is a proper subset of $cl(O(x))$. If $O(x)$ is proper for each $x \in M \subset X$, we shall say that I has proper orbits on M. Observe that in metric space (X,d) if I has d.o.d. on X, then I has proper orbits [10, 11]. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an I-contraction if, there exists $0 \leq k < 1$ such that $d(Tx,Ty) \leq kd(Ix,Iy)$ for any $x, y \in M$. If $k = 1$, then T is called I-nonexpansive. A mapping $T : M \to M$ is called (1) completely continuous if $\{x_n\}$ converges weakly to x implies that $\{Tx_n\}$ converges strongly to Tx; (2) demiclosed at 0 if for every sequence $\{x_n\} \in M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$. The mappings I and T are said to satisfy the condition (A^0) if for any sequence $\{x_n\}$ in M, $D \in C(M)$ such that $dist(x_n, D) \to 0$ and $d(Ix_n, Tx_n) \to 0$ as $n \to \infty$, there exists $y \in D$ with $Iy = Ty$, where $C(M)$ denotes the class of nonempty closed subsets of M. The set of fixed points of T (resp. I) is denoted by $F(T)$ (resp. $F(I)$). A point $x \in M$ is a common fixed (coincidence) point of I and T if $x = Ix = Tx(Ix = Tx)$. The set of coincidence points of I and T is denoted by $C(I,T)$. The pair $\{I,T\}$ is called (3) commuting if $TIx = ITx$ for all $x \in M$; (4) R-weakly commuting if for all $x \in M$ there exists $R > 0$ such that $d(ITx, TIx) \leq$R$d(Ix,Tx)$. If $R = 1$, then the maps are called weakly commuting; (5) compatible [9] if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M; (6) weakly compatible if they commute at their coincidence points, i.e.,if $ITx = TIx$ whenever $Ix = Tx$. If I and T are weakly compatible and do have a coincidence point, I and T are called [3, 10] nontrivially weakly compatible. The subset M of a linear space is called q-starshaped with $q \in M$ if the segment $[q,x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to

x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(I)$ and is both T- and I-invariant. Then T and I are called;

(7) R-subcommuting on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq \frac{R}{k} d((1-k)q + kTx, Ix)$ for each $k \in (0, 1]$. If $R = 1$, then the maps are called 1-subcommuting [7]; (8) R-subweakly commuting on M (see [8, 23]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(ITx, TIx) \leq R dist(Ix, [q, Tx])$. Clearly, R-weakly commuting, and compatible maps are weakly compatible but not conversely in general. R-subcommuting and R-subweakly commuting maps are compatible but the converse does not hold in general [11].

In 1995, Jungck and Sessa [12] extended the results of Meinardus [17], Singh [25], Habiniak [4] and Sahab, Khan and Sessa [21] to the pair of commuting maps defined on weakly compact subset of a Banach space. Latif [16], further extended these results to the setting of p-normed spaces. More recently, Shahzad [23, 24]], Hussain and Jungck [11], Hussain et al. [8], Jungck and Hussain [11] and O'Regan and Hussain [19] further extended the above-mentioned results to R-subweakly commuting and weakly compatible maps. The aim of this paper is to establish a general common fixed point theorem for compatible and weakly compatible generalized I-nonexpansive maps in the setting of locally bounded topological vector spaces and locally convex topological vector spaces. As application, we derive some results on the existence of best approximations. Our results unify and extend the results of Dotson [1, 2], Habiniak [4], Hussain and Berinde [5], Hussain and Khan [7], Hussain, O'Regan and Agarwal [8], Jungck and Sessa [12], Khan et al. [13], Khan and Khan [14], Latif [16], O'Regan and Hussain [19], Sahab et al. [21], Sahney et al. [22], Shahzad [23, 24], and Singh [25, 26].

Here, we state some useful results.

THEOREM 1.1 [3]. *Let X be a Hausdorff topological space, and I, T be continuous and nontrivially weakly compatible self-maps of X. Then there exists a point z in X such that $Iz = Tz = z$, provided T satisfies following condition*

(C) $\quad A \cap F(T) \neq \emptyset$ *for any nonempty T-invariant closed set $A \subset X$.*

The next theorem gives conditions under which condition (C) is satisfied.

THEOREM 1.2 ([10], Theorem 3.1). *Let X be a Hausdorff topological space and T be a continuous self-map of X. If T has relatively compact proper orbits then T satisfies condition (C).*

2. Common fixed point and approximation results

The following recent result will be needed in the sequel.

THEOREM 2.1 [11]. *Let M be a subset of a metric space (X, d), and I and T be self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete and I, T satisfy for all $x, y \in M$ and $0 \leq h < 1$ the condition*

$$(2.1) \quad d(Tx, Ty) \leq h \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\}$$

Then I and T have a unique coincidence point in M.

Throughout this section, we shall assume that X^* separates points of a p-normed space X whenever weak topology is under consideration.

THEOREM 2.2. *Let I and T be self-maps on a q-starshaped subset M of a p-normed space X. Assume that T satisfies condition (C), $clT(M) \subset I(M)$, $q \in F(I)$ and I is affine. Suppose that I and T are continuous, and satisfy*

$$(2.2) \quad \|Tx - Ty\|_p \leq max \left\{ \begin{array}{c} \|Ix - Iy\|_p, dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ dist(Ix, [Ty, q]), dist(Iy, [Tx, q]) \end{array} \right\}$$

for all $x, y \in M$. Then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds;

(i) *$clT(M)$ is compact and I and T are compatible,*
(ii) *M is complete and bounded, T is a compact map and I and T are compatible,*
(iii) *M is complete and bounded, I and T satisfy condition (A^0) and I and T are weakly compatible,*
(iv) *X is complete, M is weakly compact, $I - T$ is demiclosed at 0 and I and T are weakly compatible,*
(v) *X is complete, M is weakly compact, I and T are completely continuous and I and T are weakly compatible.*

Proof. Define $T_n : M \to M$ by $T_n x = (1 - k_n)q + k_n Tx$ for some q and all $x \in M$ and a fixed sequence of real numbers $k_n \in (0, 1)$ converging to 1. Then, for each n, $clT_n(M) \subset I(M)$ as M is q-starshaped, $clT(M) \subset I(M)$, I is affine and $Iq = q$. By (2.2),

$$\begin{aligned} \|T_n x - T_n y\|_p &= (k_n)^p \|Tx - Ty\|_p \\ &\leq (k_n)^p max\{\|Ix - Iy\|_p, dist(Ix, [Tx, q]), dist(Iy, [Ty, q]), \\ &\quad dist(Ix, [Ty, q]), dist(Iy, [Tx, q])\} \\ &\leq (k_n)^p max\{\|Ix - Iy\|_p, \|Ix - T_n x\|_p, \|Iy - T_n y\|_p, \\ &\quad \|Ix - T_n y\|_p, \|Iy - T_n x\|_p\}, \end{aligned}$$

for each $x, y \in M$.

(i) Since $clT(M)$ is compact, $clT_n(M)$ is also compact and hence complete. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in M$ such that $Ix_n = T_nx_n$. The compactness of $cl(T(M)$ implies that there exists a subsequence $\{Tx_m\}$of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Since $k_m \to 1$, $Ix_m = (1-k_m)q + k_mTx_m$ converges to y. Since T and I are continuous, then $TIx_m \to Ty$ and $ITx_m \to Iy$ as $m \to \infty$. By the compatibility of I and T, we obtain $0 = \lim_{m\to\infty} \|ITx_m - TIx_m\|_p = \|Iy - Ty\|_p$. Thus $Iy = Ty$. Hence the pair $\{I, T\}$ is nontrivially compatible. Theorem 1.1 guarantees that $M \cap F(I) \cap F(T) \neq \emptyset$.

(ii) As in (i), there is a unique $x_n \in M$ such that $T_nx_n = Ix_n$. As T is compact and $\{x_n\}$ being in M is bounded so $\{Tx_n\}$ has a subsequence $\{Tx_m\}$ such that $\{Tx_m\} \to z$ as $m \to \infty$. Then the definition of T_mx_m implies $Ix_m \to z$. So by the continuity of T and I, $TIx_m \to Tz$ and $ITx_m \to Iz$ as $m \to \infty$. By the compatibility of I and T, we obtain $Iz = Tz$. Hence the pair $\{I, T\}$ is nontrivially compatible. Theorem 1.1 guarantees that $M \cap F(I) \cap F(T) \neq \emptyset$.

(iii) As in (i) there exists $x_n \in M$ such that $Ix_n = T_nx_n$. But M is bounded, so $\|Ix_n - Tx_n\|_p = \|((1-k_n)q + k_nTx_n) - Tx_n\|_p \leq (1-k_n)^p(\|q\|_p + \|Tx_n\|_p) \to 0$ as $n \to \infty$. By condition (A^0), $Ix_0 = Tx_0$ for some $x_0 \in M$. Hence the pair $\{I, T\}$ is nontrivially weakly compatible. Theorem 1.1 guarantees that $M \cap F(I) \cap F(T) \neq \emptyset$.

(iv) Since M is weakly compact and hence complete, then $cl(T_n(M))$ is complete. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in M$ such that $Ix_n = T_nx_n$. The weak compactness of M implies that there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ weakly as $m \to \infty$. Since $\{x_m\}$ is bounded, $k_m \to 1$, so $\|(Ix_m - Tx_m\|_p = \|((1-k_m)q + k_mTx_m) - Tx_m\|_p \leq (1-k_m)^p(\|q\|_p + \|Tx_m\|_p)$ converges to 0. Since $(I-T)$ is demiclosed at 0 so $(I-T)y = 0$ and hence $Iy = Ty$. Thus the pair $\{I, T\}$ is nontrivially weakly compatible and the conclusion follows from Theorem 1.1.

(v) As in (iv), we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$. Since I and T are completely continuous, then $Ix_m \to Iy$ and $Tx_m \to Ty$ as $m \to \infty$. Since $k_m \to 1$, then $Ix_m = T_mx_m = k_mTx_m + (1-k_m)q \to Ty$ as $m \to \infty$. Using the uniqueness of the limit, we have $Iy = Ty$. Thus the pair $\{I, T\}$ is nontrivially weakly compatible and the conclusion follows from Theorem 1.1.

COROLLARY 2.3. *Let M be a q-starshaped subset of a p-normed space X, and I and T continuous self-maps of M. Suppose that I is affine with $q \in F(I)$, $clT(M) \subset I(M)$ and $clT(M)$ is compact. If T has d.o.d., the pair $\{I, T\}$ is compatible and satisfy (2.2) for all $x, y \in M$, then $M \cap F(T) \cap F(I) \neq \emptyset$.*

Proof. Since T has d.o.d, T has proper orbits [10]. As $clT(M))$ is compact, T has relatively compact orbits. Therefore by Theorem 1.2, T satisfies condition (C). The result now follows by Theorem 2.2(i).

REMARK 2.4. Theorem 2.2 and Corollary 2.3 extend and improve Theorems 1 and 2 of Dotson [1], Theorem 4 of Habiniak [4], Theorem 2.3 and Corollary 2.4 of Jungck and Hussain [11], Theorem 6 of Jungck and Sessa [12], Theorem 2.4 of O'Regan and Hussain [19], Theorem 2.2 of Shahzad [24], and corresponding results in [14, 16, 21, 23, 25].

The following result extends Theorem 3 of [21], Theorem 8 of [4], and the main results in [14, 16, 17, 25].

THEOREM 2.5. *Let M be subset of a p-normed space X and let $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that T satisfies condition (C), $I(P_M(u)) = P_M(u)$ and the pair $\{I, T\}$ is continuous and compatible on $P_M(u)$ and satisfy for all $x \in P_M(u) \cup \{u\}$,*

$$(2.3)\qquad \|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ max\{\|Ix - Iy\|_p, dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), & \\ dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\} & \text{if } y \in P_M(u), \end{cases}$$

If $P_M(u)$ is closed, q-starshaped with $q \in F(I)$, I is affine and $clT(P_M(u))$ is compact then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in P_M(u)$. Then $\|x - u\|_p = dist(u, M)$. Note that for any $k \in (0, 1)$,

$$\|ku + (1-k)x - u\|_p = (1-k)^p \|x - u\|_p < dist(u, M).$$

It follows that the line segment $\{ku + (1-k)x : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also since $Ix \in P_M(u)$, $u \in F(T) \cap F(I)$ and T and I satisfy (2.3), we have

$$\|Tx - u\|_p = \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = \|Ix - u\|_p = dist(u, M).$$

Thus $Tx \in P_M(u)$. Theorem 2.2(i) further guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

The following result provides a non-locally convex space analogue of Theorem 3.3 [7] for more general class of maps.

THEOREM 2.6. *Let M be subset of a p-normed space X and $I, T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that T satisfies condition (C). D is closed q-starshaped with $q \in F(I)$, I is affine, $clT(D)$ is compact, $I(D) = D$ and the pair $\{I, T\}$ is compatible and continuous on D and, for all $x \in D \cup \{u\}$, satisfies the following inequality,*

$$(2.4) \quad \|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ max\{\|Ix - Iy\|_p, dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), \\ dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\} & \text{if } y \in D, \end{cases}$$

If I is nonexpansive on $P_M(u) \cup \{u\}$, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$ then proceeding as in the proof of Theorem 2.5, we obtain $Tx \in P_M(u)$. Moreover, since I is nonexpansive on $P_M(u) \cup \{u\}$ and T satisfies (2.4), we obtain

$$\|ITx - u\|_p \leq \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = dist(u, M).$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $clT(D) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

REMARK 2.7. (a) It is worth to mention that approximation results similar to Theorem 2.5 and Theorem 2.6 can be obtained, using Theorem 2.2(ii)-(v) which extend and improve the corresponding results in [12, 14, 16, 17, 21, 24, 25].
(b) As an application of Theorem 2.2(i), we can prove Theorem 2.7 of [11] in the setup of p-normed space X.
(c) The results of this section hold true for the the nonlocally convex spaces, for example, the sequences spaces l_p, $0 < p < 1$ and Hardy spaces H^p, $0 < p < 1$ whose topological duals are total. When topological dual is not total the situation becomes more complicated. The topological dual of $X = L_p[0, 1]$, $0 < p < 1$, and $X = S[0, 1]$, vanish and Shauder's conjecture is still open even for these spaces (see for details [18, 20] and references therein).

3. Further results

(1) All results of the paper (Theorem 2.2-Remark 2.7) remain valid in the setup of a metrizable locally convex topological vector space (X, d), where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$ (recall that d_p is translation invariant

and satisfies $d_p(\alpha x, \alpha y) \leq (\alpha)^p d_p(x, y)$ for any scalar $\alpha \geq 0$). Consequently, Theorem 2.2-Theorem 3.3 due to Hussain and Khan [7] and corresponding results in [5, 22, 26] are improved and extended.

We define $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$ and denote by $\Im_0$ the class of closed convex subsets of X containing 0. For $M \in \Im_0$, we define $M_u = \{x \in M : d(0, x) \leq 2d(0, u)\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$.

Following result extends Theorem 8 in [4], Theorem 3.3 in [5], Theorems 2.9-2.10 in [11], Theorem 2.6 in [19], Theorem 2.3-2.4 in [23], Theorem 2.9 in [24] and many others.

THEOREM 3.1. *Let X be a metrizable locally convex space (X, d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$, and I and T be self-mappings of X with $u \in F(I) \cap F(T)$ and $M \in \Im_0$ such that $T(M_u) \subset I(M) \subset M$. Suppose that I is affine, $d(Ix, u) \leq d(x, u)$, $d(Tx, u) \leq d(Ix, u)$ for all $x \in M$, the pair $\{I, T\}$ is continuous on M and one of the following two conditions is satisfied:*

(a) *$clI(M)$ is compact,*
(b) *$clT(M)$ is compact.*

Then

(i) *$P_M(u)$ is nonempty, closed and convex,*
(ii) *$T(P_M(u)) \subset I(P_M(u)) \subset P_M(u)$ provided that $d(Ix, u) \leq d(x, u)$ for all $x \in C_M^I(u)$,*
(iii) *$P_M(u) \cap F(I) \cap F(T) \neq \emptyset$ provided that $d(Ix, u) \leq d(x, u)$ for all $x \in C_M^I(u)$, I and T satisfy condition (C), $I(P_M(u))$ is closed, the pair $\{I, T\}$ is compatible on $P_M(u)$ and satisfies for all $q \in F(I)$,*

$$d(Tx, Ty) \leq \max\{d(Ix, Iy), dist(Ix, [q, Tx]), dist(Iy, [q, Ty]), dist(Ix, [q, Ty]), dist(Iy, [q, Tx])\},$$

for all $x, y \in P_M(u)$.

Proof.

(i) Let $r = dist(u, M)$. Then there is a minimizing sequence $\{y_n\}$ in M such that $lim_n d(u, y_n) = r$. As $clI(M)$ is compact so $\{Iy_n\}$ has a convergent subsequence $\{Iy_m\}$ with $\lim_m Iy_m = x_0$ (say) in M. Now by using $d(Ix, u) \leq d(x, u)$ we get

$$r \leq d(x_0, u) = \lim_m d(Iy_m, u) \leq \lim_m d(y_m, u) = \lim_n d(y_n, u) = r.$$

Hence $x_0 \in P_M(u)$. Thus $P_M(u)$ is nonempty closed and convex. Similarly, when $clT(M)$ is compact we get same conclusion by using inequalities $d(Ix, u) \leq d(x, u)$ and $d(Tx, u) \leq d(Ix, u)$ for all $x \in M$.

(ii) Let $z \in P_M(u)$. Then $d(Tz, u) \leq d(Iz, u) = dist(u, M)$. This implies that $Tz \in P_M(u)$ and so $T(P_M(u)) \subset P_M(u)$. Also we have $I(P_M(u)) \subset P_M(u)$. Let $y \in T(P_M(u))$. Since $T(M_u) \subset I(M)$ and $P_M(u) \subset M_u$, then there exist $z \in P_M(u)$ and $x \in M$ such that $y = Tz = Ix$. Thus, we have
$d(Ix, u) = d(Tz, u) \leq d(Iz, u) \leq d(z, u) = dist(u, M)$.
Hence $x \in C_M^I(u) = P_M(u)$ and so (ii) holds.

(iii) (a) By (i) $P_M(u)$ is closed and by (ii) $P_M(u)$ is I-invariant, so by condition (C) of I, $P_M(u) \cap F(I) \neq \emptyset$. It follows that there exists $q \in P_M(u)$ such that $q \in F(I)$. By (ii), the compactness of $clI(M_u)$ implies that $clT(P_M(u))$ is compact. The conclusion now follows from Theorem 2.2(i)(which holds for metrizable locally convex space) applied to $P_M(u)$.

(iii) (b) By (i) $P_M(u)$ is closed and by (ii) $P_M(u)$ is I-invariant, so by condition (C) of I, $P_M(u) \cap F(I) \neq \emptyset$, it follows that there exists $q \in P_M(u)$ such that $q \in F(I)$. Theorem 2.2(i) further guarantees that $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.

(2) Let M be subset of a $p-$normed space X and $F = \{f_x\}_{x \in M}$ a family of functions from $[0, 1]$ into M such that $f_x(1) = x$ for each $x \in M$. The family F is said to be contractive [2, 13] if there exists a function $\phi : (0, 1) \to (0, 1)$ such that for all $x, y \in M$ and all $t \in (0, 1)$, we have $\|f_x(t) - f_y(t)\|_p \leq [\phi(t)]^p \|x - y\|_p$. The family F is said to be jointly (weakly) continuous if $t \to t_0$ in [0,1] and $x \to x_o$ ($x \to x_0$ weakly) in M, then $f_x(t) \to f_{x_0}(t_0)$ ($f_x(t) \to f_{x_0}(t_o)$ weakly) in M. We observe that if $M \in X$ is q-starshaped and $f_x(t) = (1-t)q + tx$, ($x \in M; t \in (0, 1)$), then $F = \{f_x\}_{x \in M}$ is a contractive jointly continuous and jointly weakly continuous family with $\phi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets ((see [2, 8]). Following the arguments as above and those in [8, 13], we can obtain all of the results of the paper (Theorem 2.2-Remark 2.7) provided I is assumed to be surjective, and affinity of I is replaced by $I(f_x(\alpha)) = f_{Ix}(\alpha)$ for all $x \in M$, $\alpha \in [0, 1]$, and the q-starshapedness of the set M is replaced by the property of contractivity and joint continuity or weak joint continuity. Consequently, recent results due to Hussain et al. [8], and Khan et al [13] are extended to the class of weakly compatible pair $\{I, T\}$ where T satisfies property (C).

(3) A subset M of a linear space X is said to have property (N) with respect to T [5, 8] if,

(i) $T : M \to M$,

(ii) $(1 - k_n)q + k_nTx \in M$, for some $q \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in M$.

A mapping I is said to be affine on a set M with property (N) if $I((1-k_n)q + k_nTx) = (1-k_n)Iq + k_nITx$ for each $x \in M$ and $n \in N$. All of the results of the paper (Theorem 2.3-Remark 2.7) remain valid, provided I is assumed to be surjective and the q-starshapedness of the set M is replaced by the property (N), in the setup of p-normed spaces and metrizable locally convex topological vector space(tvs) (X, d) where d is translation invariant and $d(\alpha x, \alpha y) \leq \alpha d(x, y)$, for each α with $0 < \alpha < 1$ and $x, y \in X$. Consequently, recent results due to Hussain and Berinde [5], and Hussain, O'Regan and Agarwal [8] are extended to the class of weakly compatible maps, where T satisfies property (C).

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